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ADMISSIBILITY CONDITIONS FOR DEGENERATE CYCLOTOMIC BMW ALGEBRAS

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ABSTRACT. We study admissibility conditions for the parameters of degenerate cyclotomic BMW algebras. We show that the u -admissibility condition of Ariki, Mathas and Rui is equivalent to a simple module theoretic condition.

1. INTRODUCTION

The *cyclotomic Birman–Wenzl–Murakami (BMW) algebras* are BMW analogues of cyclotomic Hecke algebras [2, 1], while the *degenerate cyclotomic BMW algebras* are BMW analogues of degenerate cyclotomic Hecke algebras [10].

The cyclotomic BMW algebras were defined by Häring–Oldenburg in [9] and have recently been studied by three groups of mathematicians: Goodman and Hauschild–Mosley [6, 7, 8, 4], Rui, Xu, and Si [14, 12], and Wilcox and Yu [16, 17, 15, 18].

Degenerate affine BMW algebras were introduced by Nazarov [11] under the name *affine Wenzl algebras*. The cyclotomic quotients of these algebras were introduced by Ariki, Mathas, and Rui in [3] and studied further by Rui and Si in [13], under the name *cyclotomic Nazarov–Wenzl algebras*. (We propose to refer to these algebras as degenerate affine (resp. degenerate cyclotomic) BMW algebras instead, to bring the terminology in line with that used for degenerate affine and cyclotomic Hecke algebras.)

A peculiar feature of the cyclotomic and degenerate cyclotomic BMW algebras is that it is necessary to impose “admissibility” conditions on the parameters entering into the definition of the algebras in order to obtain a satisfactory theory. For the cyclotomic BMW algebras, two apparently different conditions were proposed, one by Wilcox and Yu [16] and another by Rui and Xu [14]. We recently showed [5] that the two conditions are equivalent. Moreover, according to [16], admissibility is equivalent to a simple module theoretic condition: the left ideal W_2e generated by the “contraction” e in the two-strand algebra is free of the maximal possible rank.

It is natural to ask for similar results regarding the parameters of degenerate cyclotomic BMW algebras. Ariki, Mathas and Rui [3] introduced an admissibility condition (called u -admissibility) for these algebras, based on a heuristic involving the rank of the left ideal W_2e in the two-strand algebra, but up until now it has not been shown that their condition is equivalent to W_2e being free of maximal rank. In this note,

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we introduce an analogue for the degenerate cyclotomic BMW algebras of the admissibility condition of Wilcox and Yu [16], we show that this condition is equivalent to u -admissibility, and that both conditions are equivalent to W_2e being free of maximal rank.

2. DEFINITIONS

Fix a positive integer n and a commutative ring R with multiplicative identity. Let $\Omega = \{\omega_a : a \geq 0\}$ be a sequence of elements of R .

Definition 2.1 (Nazarov [11]). The *degenerate affine BMW algebra* $W_n^{\text{aff}} = W_n^{\text{aff}}(\Omega)$ is the unital associative R -algebra with generators $\{s_i, e_i, x_j : 1 \leq i < n \text{ and } 1 \leq j \leq n\}$ and relations:

- (1) (Involutions) $s_i^2 = 1$, for $1 \leq i < n$.
- (2) (Affine braid relations)
 - (a) $s_i s_j = s_j s_i$ if $|i - j| > 1$,
 - (b) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $1 \leq i < n - 1$,
 - (c) $s_i x_j = x_j s_i$ if $j \neq i, i + 1$.
- (3) (Idempotent relations) $e_i^2 = \omega_0 e_i$, for $1 \leq i < n$.
- (4) (Commutation relations)
 - (a) $s_i e_j = e_j s_i$, if $|i - j| > 1$,
 - (b) $e_i e_j = e_j e_i$, if $|i - j| > 1$,
 - (c) $e_i x_j = x_j e_i$, if $j \neq i, i + 1$,
 - (d) $x_i x_j = x_j x_i$, for $1 \leq i, j \leq n$.
- (5) (Skein relations) $s_i x_i - x_{i+1} s_i = e_i - 1$, and $x_i s_i - s_i x_{i+1} = e_i - 1$, for $1 \leq i < n$.
- (6) (Unwrapping relations) $e_1 x_1^a e_1 = \omega_a e_1$, for $a > 0$.
- (7) (Tangle relations)
 - (a) $e_i s_i = e_i = s_i e_i$, for $1 \leq i \leq n - 1$,
 - (b) $s_i e_{i+1} e_i = s_{i+1} e_i$, and $e_i e_{i+1} s_i = e_i s_{i+1}$, for $1 \leq i \leq n - 2$,
 - (c) $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$, and $s_{i+1} e_i e_{i+1} = s_i e_{i+1}$, for $1 \leq i \leq n - 2$.
- (8) (Untwisting relations) $e_{i+1} e_i e_{i+1} = e_{i+1}$, and $e_i e_{i+1} e_i = e_i$, for $1 \leq i \leq n - 2$.
- (9) (Anti-symmetry relations) $e_i(x_i + x_{i+1}) = 0$, and $(x_i + x_{i+1})e_i = 0$, for $1 \leq i < n$.

Definition 2.2 (Ariki, Mathas, Rui [3]). Fix an integer $r \geq 1$ and elements u_1, \dots, u_r in R . The *degenerate cyclotomic BMW algebra* $W_{r,n} = W_{r,n}(u_1, \dots, u_r)$ is the R -algebra

$$W_n^{\text{aff}}(\Omega) / \langle (x_1 - u_1) \dots (x_1 - u_r) \rangle.$$

Note that, due to the symmetry of the relations, W_n^{aff} has a unique R -linear algebra involution $*$ (that is, an algebra anti-automorphism of order 2) such that $e_i^* = e_i$, $s_i^* = s_i$, and $x_i^* = x_i$ for all i . The involution passes to cyclotomic quotients.

Lemma 2.3 (see [3], Lemma 2.3). *In the degenerate affine BMW algebra W_n^{aff} , for $1 \leq i < n$ and $a \geq 1$, one has*

$$(2.1) \quad s_i x_i^a = x_{i+1}^a s_i + \sum_{b=1}^a x_{i+1}^{b-1} (e_i - 1) x_i^{a-b}.$$

Taking $i = 1$ in Lemma 2.3, pre- and post-multiplying by e_1 and simplifying using the relations gives:

$$(2.2) \quad \omega_a e_1 = (-1)^a \omega_a e_1 + \sum_{b=1}^a (-1)^{b-1} \omega_{b-1} \omega_{a-b} e_1 + \sum_{b=1}^a (-1)^b \omega_{a-1} e_1$$

For a odd, this gives

$$(2.3) \quad 2\omega_a e_1 = \sum_{b=1}^a (-1)^{b-1} \omega_{b-1} \omega_{a-b} e_1 - \omega_{a-1} e_1,$$

which is Corollary 2.4 in [3]. As noted in [3], the identity derived from (2.2) in case a is even is a tautology.

Consider the cyclotomic algebra $W_{r,n}(u_1, \dots, u_r)$, and let a_j denote the signed elementary symmetric function in u_1, \dots, u_r , namely, $a_j = (-1)^{r-j} \varepsilon_{r-j}(u_1, \dots, u_r)$. Thus, in the cyclotomic algebra, we have the relation $\sum_{j=0}^r a_j x_1^j = 0$. Multiplying by x_1^a for an arbitrary $a \geq 0$ and pre- and post-multiplying by e_1 gives

$$(2.4) \quad \sum_{j=0}^r a_j \omega_{j+a} e_1 = 0.$$

Corollary 2.4. *Consider the cyclotomic algebra $W_{r,n}(u_1, \dots, u_r)$. If e_1 is not a torsion element over R , then we have:*

- (1) $2\omega_a = \sum_{b=1}^a (-1)^{b-1} \omega_{b-1} \omega_{a-b} - \omega_{a-1}$, for all odd $a \geq 1$, and
- (2) $\sum_{j=0}^r a_j \omega_{j+a} = 0$, for all $a \geq 0$.

Definition 2.5. Say that the parameters ω_a ($a \geq 0$) and u_1, \dots, u_r are *weakly admissible*, or that the ground ring R is *weakly admissible*, if the relations of Corollary 2.4 hold.

Weak admissibility is a non-triviality condition for the cyclotomic algebras; if the ground ring is a field, and weak admissibility fails, then $e_1 = 0$, and the cyclotomic algebra reduces to a specialization of the degenerate cyclotomic Hecke algebra, see [3], pages 60–61.

In the following, we use the notation $\delta_{(P)} = 1$ if P is true and $\delta_{(P)} = 0$ if P is false.

Lemma 2.6. *In the degenerate affine BMW algebra, for $a \geq 1$, we have*

$$(2.5) \quad s_1 x_1^a e_1 = (-1)^a x_1^a e_1 + \sum_{b=1}^a (-1)^{b-1} \omega_{a-b} x_1^{b-1} e_1 - \delta_{(a \text{ is odd})} x_1^{a-1} e_1.$$

Proof. Take $i = 1$ in equation (2.1). Post-multiply by e_1 , and simplify, using the relations. \square

3. u -ADMISSIBILITY

The definition of u -admissibility is motivated by Theorem 3.2 below, which is essentially contained in [3], although not explicitly stated there.

Lemma 3.1. *Let R be any ground ring with parameters ω_a for $a \geq 0$ and u_1, \dots, u_r . Let $W_{2,R}$ denote the two strand degenerate cyclotomic BMW algebra defined over R . Then*

- (1) *The left ideal $W_{2,R} e_1$ equals the R -span of $\{e_1, x_1 e_1, \dots, x_1^{r-1} e_1\}$.*
- (2) *$W_{2,R}$ is spanned over R by $\{x_1^a e_1 x_1^b, x_1^a x_2^b s_1, x_1^a x_2^b : 0 \leq a, b \leq r-1\}$*

Proof. Using Lemma 2.3, and the defining relations of the algebra, one sees that the span of $\{e_1, x_1 e_1, \dots, x_1^{r-1} e_1\}$ is invariant under left multiplication by the generators x_1, e_1, s_1 , and that $x_2 x_1^a e_1 = -x_1^{a+1} e_1$. This proves part (1). Part (2) is similar, see [3], Proposition 2.15. \square

Theorem 3.2 ([3]). *Let F be a field of characteristic $\neq 2$, with parameters ω_a for $a \geq 0$ and u_1, \dots, u_r . Assume that the u_i are distinct and $u_i + u_j \neq 0$ for all i, j . Let $W = W_{2,F}$ be the degenerate cyclotomic BMW algebra defined over F with parameters ω_a for $a \geq 0$ and u_1, \dots, u_r . Then the following conditions are equivalent:*

- (1) *$\{e_1, x_1 e_1, \dots, x_1^{r-1} e_1\} \subseteq W e_1$ is linearly independent over F , and $e_1 W e_1 \neq 0$.*
- (2) *For all $a \geq 0$, $\omega_a = \sum_{i=1}^r \gamma_i u_i^a$, where*

$$(3.1) \quad \gamma_i = (2u_i - (-1)^r) \prod_{j \neq i} \frac{u_i + u_j}{u_i - u_j},$$

and some ω_a is non-zero.

- (3) *W admits a module M with basis $\{v_0, x_1 v_0, \dots, x_1^{r-1} v_0\}$ such that $e_1 M = F v_0$.*

Proof. The implication (1) \implies (3) is obvious.

Assume condition (3). We have $v_0 = e_1 m$ for some $m \in M$, so $e_1 x^j v_0 = e_1 x^j e_1 m = \omega_j e_1 m = \omega_j v_0$ for $1 \leq j \leq r-1$. Moreover, some $\omega_j \neq 0$ since $e_1 M \neq (0)$. Define $p_i \in W_{1,F}$ by $p_i = \prod_{j \neq i} \frac{x_1 - u_j}{u_i - u_j}$. Then $p_i^2 = p_i$, $\sum_i p_i = 1$, and $x_1 p_i = u_i p_i$. Define $m_i \in M$ by $m_i = p_i v_0$. Then $m_i \neq 0$ by the assumed linear independence of $\{v_0, x_1 v_0, \dots, x_1^{r-1} v_0\}$, $x_1 m_i = u_i m_i$, and $\sum_i m_i = (\sum_i p_i) v_0 = v_0$. It follows that $\{m_1, \dots, m_r\}$ is linearly independent, since the m_i are eigenvectors for x_1 with distinct eigenvalues. Define κ_j and $c_{i,j}$ in F by $e_1 m_j = \kappa_j v_0 = \kappa_j \sum_i m_i$, and $s_1 m_j = \sum_i c_{i,j} m_i$. (It will be shown that $\kappa_j = \gamma_j$, where γ_j is defined above.) Note that $e_1 M \neq (0)$ implies that $\kappa_j \neq 0$ for some j .

The argument continues as in the proof of Theorem 3.2 in [3], pp. 65–67. Apply the identity $x_1 s_1 - s_1 x_2 - e_1 + 1 = 0$ to m_j to derive a formula for $c_{i,j}$,

$$c_{i,j} = (\kappa_j - \delta_{i,j}) / (u_i + u_j).$$

Next apply the identity $e_1 = s_1 e_1$ to m_i to get

$$(3.2) \quad \kappa_i \sum_{j=1}^d m_j = e_1 m_i = s_1 e_1 m_i = \kappa_i \sum_{j=1}^d \left\{ \frac{\kappa_j - 1}{2u_j} + \sum_{k \neq j} \frac{\kappa_k}{u_j + u_k} \right\} m_j,$$

for $i = 1, \dots, r$. Since at least one κ_i is non-zero, matching coefficients in (3.2) gives the equations

$$(3.3) \quad \sum_{k=1}^d \frac{\kappa_k}{u_j + u_k} = 1 + \frac{1}{2u_j},$$

for $j = 1, \dots, r$. Now it is shown in [3], page 66, that the unique solution to this system of equations is $\kappa_j = \gamma_j$ for $1 \leq j \leq r$. Finally, we have

$$(3.4) \quad \omega_j v_0 = e_1 x_1^j v_0 = e_1 x_1^j \left(\sum_i m_i \right) = e_1 \sum_i u_i^j m_i = \left(\sum_i u_i^j \gamma_i \right) v_0.$$

This shows (3) \implies (2).

Finally, (2) \implies (1) by Theorem A of [3], namely assuming (2), W has an R -basis that includes $\{e_1, x_1 e_1, \dots, x_1^{r-1} e_1\}$, so the latter set is linearly independent. \square

The elements γ_j appearing in Theorem 3.2 are rational functions in u_1, \dots, u_r with singularities at $u_i = u_j$, but it is shown in [3] that the elements $\sum_i \gamma_i u_i^a$ are polynomials in u_1, \dots, u_r , as follows:

Let $\mathbf{u}_1, \dots, \mathbf{u}_r$ and t be algebraically independent indeterminants over \mathbb{Z} . Define symmetric polynomials $q_a(\mathbf{u})$ in $\mathbf{u}_1, \dots, \mathbf{u}_r$ by

$$\prod_{i=1}^r \frac{1 + \mathbf{u}_i t}{1 - \mathbf{u}_i t} = \sum_{a \geq 0} q_a(\mathbf{u}) t^a.$$

The polynomials q_a are known as *Schur q -functions*. Let $\gamma_j(\mathbf{u})$ be defined by (3.1) with u_i replaced by \mathbf{u}_i . Moreover, let $\eta_a(\mathbf{u}) = \sum_{i=1}^r \gamma_i(\mathbf{u}) \mathbf{u}_i^a$ for $a \geq 0$. Then ([3], Lemma 3.5)

$$(3.5) \quad \eta_a(\mathbf{u}) = q_{a+1}(\mathbf{u}) - \frac{1}{2}(-1)^r q_a(\mathbf{u}) + \frac{1}{2} \delta_{a,0}.$$

In particular the η_a are polynomials in $\mathbf{u}_1, \dots, \mathbf{u}_r$.

Corollary 3.3. *Let F be a field of characteristic $\neq 2$, with parameters ω_a for $a \geq 0$ and u_1, \dots, u_r . Assume that the u_i are distinct and $u_i + u_j \neq 0$ for all i, j . Let $W = W_{2,F}$ be the degenerate cyclotomic BMW algebra defined over F with parameters ω_a for $a \geq 0$ and u_1, \dots, u_r . If $\{e_1, x_1 e_1, \dots, x_1^{r-1} e_1\} \subseteq W e_1$ is linearly independent over F , and $e_1 W e_1 \neq 0$, then*

$$(3.6) \quad \omega_a = q_{a+1}(u_1, \dots, u_r) - \frac{1}{2}(-1)^r q_a(u_1, \dots, u_r) + \frac{1}{2} \delta_{a,0}.$$

This motivates the following definition, which makes sense for arbitrary u_1, \dots, u_r :

Definition 3.4 ([3]). Let R be a commutative ring with parameters ω_a ($a \geq 0$) and u_1, \dots, u_r . Suppose that 2 is invertible in R . Say that the parameters are u -admissible if

$$(3.7) \quad \omega_a = q_{a+1}(u_1, \dots, u_r) + \frac{1}{2}(-1)^{r-1} q_a(u_1, \dots, u_r) + \frac{1}{2} \delta_{a,0}$$

for all $a \geq 0$.

4. ADMISSIBILITY

We fix a ground ring R with parameters ω_a ($a \geq 0$) and u_1, \dots, u_r . We consider the two strand degenerate cyclotomic BMW algebra over R , $W = W_{2,r}(u_1, \dots, u_r)$ and we write $e = e_1$, $s = s_1$, and $x = x_1$.

Lemma 4.1. *Suppose that $\{e, xe, \dots, x^{r-1}e\}$ is linearly independent over R . Then the parameters ω_a ($a \geq 0$) and u_1, \dots, u_r are weakly admissible and satisfy the following relations:*

$$(4.1) \quad \sum_{\mu=0}^{r-j-1} \omega_{\mu} a_{\mu+j+1} = -2\delta_{(r-j \text{ is odd})} a_j + \delta_{(j \text{ is even})} a_{j+1},$$

for $0 \leq j \leq r-1$.

Proof. Since $\{e, xe, \dots, x^{r-1}e\}$ is assumed linearly independent over R , in particular e is not a torsion element over R , and hence R is weakly admissible by Corollary 2.4.

If $r = 1$, (4.1) reduces to the single equation $\omega_0 + 2a_0 - 1 = 0$, which follows from $(sx + xs + 1 - e)e = 0$, together with $x = u_1 = -a_0$ and $se = e$. Assume $r \geq 2$. We have

$$0 = (sx - x_2s + 1 - e)x^{r-1}e = (sx + xs + 1 - e)x^{r-1}e.$$

Apply the identity $x(x^{r-1}e) = -\sum_{j=0}^{r-1} a_j x^j e$ as well as the identity (2.5) and simplify. This gives:

$$\begin{aligned} 0 = & -a_0 e - \sum_{j=1}^{r-1} (-1)^j a_j x^j e + \sum_{\substack{0 \leq j \leq r-2 \\ j \text{ even}}} a_{j+1} x^j e + \sum_{j=0}^{r-2} (-1)^j \left(\sum_{\mu=0}^{r-j-2} \omega_{\mu} a_{\mu+j+1} \right) x^j e \\ & + (-1)^r \sum_{j=0}^{r-1} a_j x^j e - \delta_{(r \text{ is even})} x^{r-1} e + \sum_{j=1}^{r-1} \omega_{r-1-j} x^j e \\ & + x^{r-1} e - \omega_{r-1} e, \end{aligned}$$

where the three lines of the display correspond to evaluation of $sxx^{r-1}e$, $sxx^{r-1}e$, and $(1-e)x^{r-1}e$. Because $\{e, xe, \dots, x^{r-1}e\}$ is assumed to be linearly independent, the coefficient of $x^j e$ is zero for each j , $0 \leq j \leq r-1$. Extracting the coefficients yields (4.1). Here one has to treat the three cases $j = 0$, $1 \leq j \leq r-2$, and $j = r-1$ separately, but the result in all three cases is the same. \square

Definition 4.2. Say that the parameters ω_a ($a \geq 0$) and u_1, \dots, u_r are *admissible* (or that the ground ring R is admissible) if the relations (4.1) hold for $0 \leq j \leq r-1$ and $\sum_{\mu=0}^r a_{\mu} \omega_{\mu+a} = 0$ holds for all $a \geq 0$.

Remark 4.3. Admissibility is analogous to the admissibility condition of Wilcox and Yu for the cyclotomic BMW algebras [16]. Our terminology differs from that in [3], where admissibility means essentially what we have called weak admissibility.

Lemma 4.4. *There exist universal polynomials $H_a(\mathbf{u}_1, \dots, \mathbf{u}_r)$ ($a \geq 0$), symmetric in $\mathbf{u}_1, \dots, \mathbf{u}_r$, such that whenever R is an admissible ring, one has*

$$(4.2) \quad \omega_a = H_a(u_1, \dots, u_r)$$

for $a \geq 0$.

Proof. The system of relations (4.1) is a unitriangular linear system of equation for the variables $\omega_0, \dots, \omega_{r-1}$. In fact, if we list the equations in reverse order then the matrix of coefficients is

$$\begin{bmatrix} 1 & & & & \\ a_{r-1} & 1 & & & \\ a_{r-2} & a_{r-1} & 1 & & \\ \vdots & & \ddots & \ddots & \\ a_1 & a_2 & \dots & a_{r-1} & 1 \end{bmatrix}.$$

Solving the system for $\omega_0, \dots, \omega_{r-1}$ gives these quantities as polynomial functions of a_0, \dots, a_{r-1} , thus symmetric polynomials in u_1, \dots, u_r . The relations $\sum_{j=0}^r a_j \omega_{j+m} = 0$, for all $m \geq 0$ yield (4.2) for $a \geq r$. \square

5. EQUIVALENCE OF ADMISSIBILITY CONDITIONS

In this section we will show that admissibility and u -admissibility are equivalent (assuming that 2 is invertible in the ground ring, so that u -admissibility makes sense.)

First, we will obtain the polynomials H_a of Lemma 4.4 explicitly in terms of the Schur q -functions. Considering the generating function for the Schur q -functions,

$$(5.1) \quad \prod_{i=1}^r \frac{1 + \mathbf{u}_i t}{1 - \mathbf{u}_i t} = \sum_{a \geq 0} q_a(\mathbf{u}) t^a,$$

we have

$$(5.2) \quad \left(\prod_{i=1}^r (1 - \mathbf{u}_i t) \right) \left(\sum_{a \geq 0} q_a(\mathbf{u}) t^a \right) = \prod_{i=1}^r (1 + \mathbf{u}_i t).$$

Taking into account that $q_0(\mathbf{u}) = 1$, we also have

$$(5.3) \quad \left(\prod_{i=1}^r (1 - \mathbf{u}_i t) \right) \left(\sum_{a \geq 1} q_a(\mathbf{u}) t^a \right) = \prod_{i=1}^r (1 + \mathbf{u}_i t) - \prod_{i=1}^r (1 - \mathbf{u}_i t).$$

Matching coefficients in (5.2) and writing in terms of the signed elementary symmetric functions $a_i(\mathbf{u})$ gives

$$(5.4) \quad \sum_{\mu=0}^{r-j-1} q_\mu(\mathbf{u}) a_{\mu+j+1}(\mathbf{u}) = (-1)^{r-j-1} a_{j+1}(\mathbf{u}), \quad \text{for } 0 \leq j \leq r-1,$$

and, moreover,

$$(5.5) \quad \sum_{\mu=0}^r a_\mu(\mathbf{u}) q_{\mu+a}(\mathbf{u}) = \begin{cases} (-1)^r a_0(\mathbf{u}) & \text{if } a = 0, \\ 0 & \text{if } a \geq 1. \end{cases}$$

Doing the same with (5.3) yields

$$(5.6) \quad \sum_{\mu=0}^{r-j-1} q_{\mu+1}(\mathbf{u}) a_{\mu+j+1}(\mathbf{u}) = -2\delta_{(r-j \text{ is odd})} a_j(\mathbf{u}), \quad \text{for } 0 \leq j \leq r-1.$$

If we set

$$\eta_a(\mathbf{u}) = q_{a+1}(\mathbf{u}) + \frac{1}{2}(-1)^{r-1} q_a(\mathbf{u}) + \frac{1}{2}\delta_{a,0},$$

then, using (5.4) and (5.6), we get

$$(5.7) \quad \sum_{\mu=0}^{r-j-1} \eta_{\mu}(\mathbf{u}) a_{\mu+j+1}(\mathbf{u}) = -2\delta_{(r-j \text{ is odd})} a_j(\mathbf{u}) + \delta_{(j \text{ is even})} a_{j+1}(\mathbf{u}),$$

for $0 \leq j \leq r-1$. From (5.5), we obtain

$$(5.8) \quad \sum_{\mu=0}^r a_{\mu}(\mathbf{u}) \eta_{\mu+a}(\mathbf{u}) = 0, \quad \text{for all } a \geq 0.$$

Lemma 5.1. *Let R be commutative ring in which 2 is invertible. Let ω_a ($a \geq 0$) and u_1, \dots, u_r be parameters in R . Then the parameters are admissible, if, and only if, they are u -admissible.*

Proof. By definition, the parameters are u -admissible if $\omega_a = \eta_a(u_1, \dots, u_r)$ for all $a \geq 0$. It follows from (5.7) and (5.8) that u -admissible parameters are admissible.

On the other hand, if the parameters are admissible, then the relations (4.1) for $0 \leq j \leq r-1$ and $\sum_{\mu=0}^r a_{\mu} \omega_{\mu+a} = 0$ for $a \geq 0$ uniquely determine the ω_a for all $a \geq 0$ as symmetric polynomial functions of u_1, \dots, u_r . But according to (5.7) and (5.8), the elements $\eta_a(u_1, \dots, u_r)$ satisfy the same relations. Hence $\omega_a = \eta_a(u_1, \dots, u_r)$ for $a \geq 0$, so the parameters are u -admissible. \square

Theorem 5.2. *Let R be a commutative ring with parameters ω_a ($a \geq 0$) and u_1, \dots, u_r . Suppose that 2 is invertible in R . Consider the two strand degenerate cyclotomic BMW algebra over R , $W = W_{2,r}(u_1, \dots, u_r)$. The following are equivalent:*

- (1) $\{e_1, x_1 e_1, \dots, x_1^{r-1} e_1\}$ is linearly independent over R .
- (2) $\{x_1^a e_1 x_1^b, x_1^a x_2^b s_1, x_1^a x_2^b : 0 \leq a, b \leq r-1\}$ is linearly independent over R .
- (3) The parameters are admissible.
- (4) The parameters are u -admissible.

Proof. Lemma 4.1 gives (1) \implies (3). Lemma 5.1 gives (3) \iff (4). The implication (4) \implies (2) is part of the main result (Theorem A) of [3]. Finally (2) \implies (1) is trivial. \square

If the equivalent conditions of the theorem hold, then the sets in (1) and (2) are R -bases of $W_{2,R}e_1$, respectively of $W_{2,R}$, since they are spanning by Lemma 3.1. If R is an integral domain the conditions are equivalent to: (1') $W_{2,R}e_1$ is free over R of rank r , respectively (2') $W_{2,R}$ is free over R of rank $3r^2$.

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